Mechanics of Peridynamic Membranes

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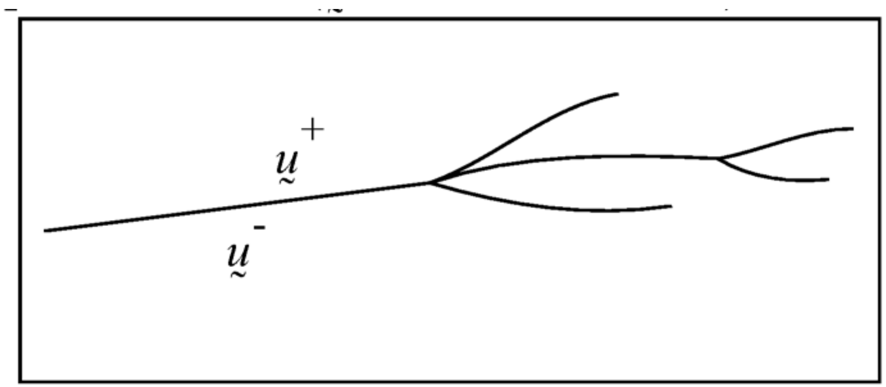
A problem with the classical theory

- PDEs don't apply when a crack or other discontinuity appears.
  - This has led to the special techniques of fracture mechanics…
  - … which are not always satisfactory.
- Purpose of the <u>peridynamic</u> model:
  - *Reformulate the basic equations so that they hold everywhere in a body regardless of discontinuities.*



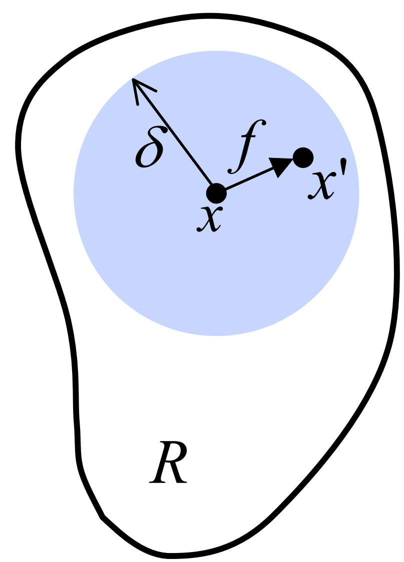
Peridynamic* model

- Replace the $\nabla \cdot \sigma$ term in the equation of motion:

$$\rho \ddot{u}(x,t) = \int_R f(u'-u, x'-x)\,dV' + b(x,t)$$

- Note the similarity to molecular dynamics.
- f is the force that x' exerts on x per unit volume squared, dependent on:
  - relative position in the reference configuration,
  - relative displacement,
  - (will consider history dependence later).
- **Not** obtainable by applying the divergence theorem to the classical PDE.
- Convenient to assume f vanishes outside some <u>horizon</u> d.
- Require:

$$f(-\eta,-\xi) = -f(\eta,\xi) \qquad\qquad f(\eta,\xi)\times(\eta+\xi) = 0$$



* From the Greek "near" + "force"

Microelastic materials

- A body is microelastic if f is derivable from a scalar **micropotential** w, i.e.,

$$f(\eta,\xi) = \frac{\partial w}{\partial \eta}(\eta,\xi)$$

$$\eta = u' - u \qquad \xi = x' - x$$

- Interactions ("bonds") can be thought of as elastic (possibly nonlinear) springs.
- Elastic energy is stored reversibly:

$$\dot{\Phi} = \int_R b \cdot \dot{u}\, dV$$

  – where the strain energy density is

$$W(x) = \frac{1}{2}\int_R w(u' - u, x' - x)\, dV'$$

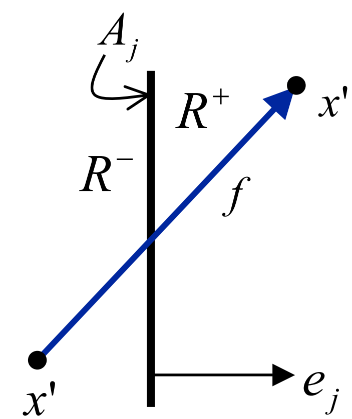
  – and the total strain energy is

$$\Phi = \int_R W(x)\, dV$$

Relation to classical theory

- For a given microelastic material with micropotential *w*, we can *define* a **classical hyperelastic** material through

$$\hat{W}(F) = \frac{1}{2}\int_R w((F-1)x, x'-x)\,dV'$$



- Can *define* a stress-like quantity

$$\sigma_{ij}(x) = \lim_{A_j \to 0}\left\{ \frac{1}{A_j}\int_{R^+}\int_{R^-} f_i(u''-u', x''-x')\,dV''dV' \right\}$$

but this is meaningful only for homogeneous deformations.

- Can show that the peridynamic equation of motion “converges to” the classical version in the limit $\delta \to 0$.

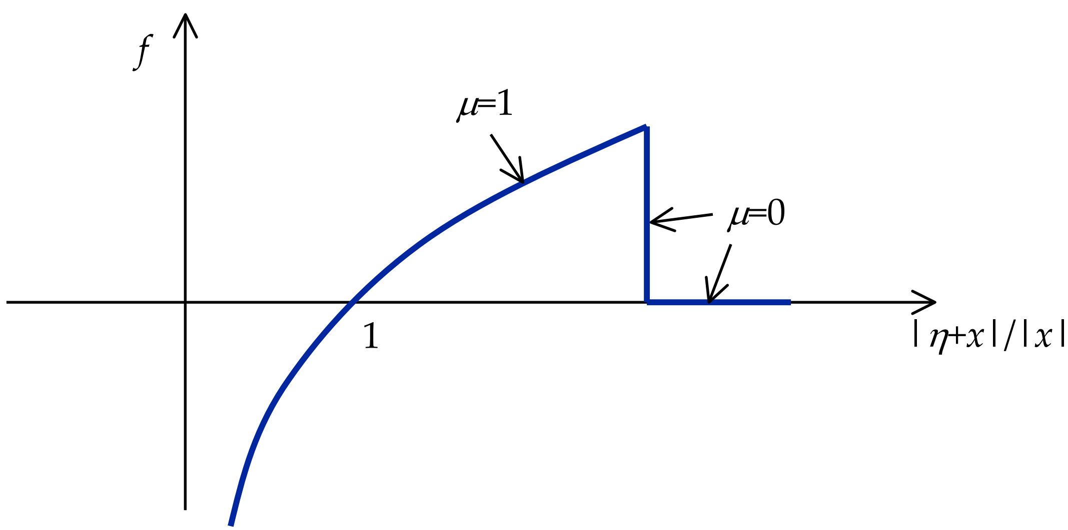
Damage

- Damage is introduced at the bond level:

$$\bar{f}(\eta,\xi,x,t) = f(\eta,\xi)\mu(\xi,x,t)$$

where μ =1 for an intact bond, 0 for a broken bond.

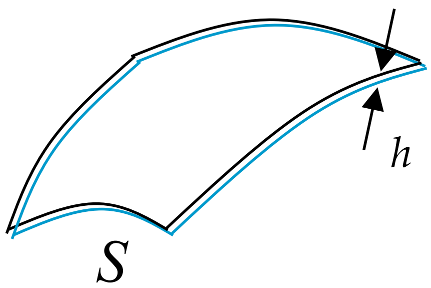
- Bond breakage occurs irreversibly according to some criterion such as exceeding a prescribed critical stretch.



Microelastic membranes

- Equation of motion for a membrane with thickness h:

$$\rho\ddot{u} = h\int_S f(u'-u, x'-x)\,dV' + b$$



- Prototype constitutive model for a peridynamic membrane:

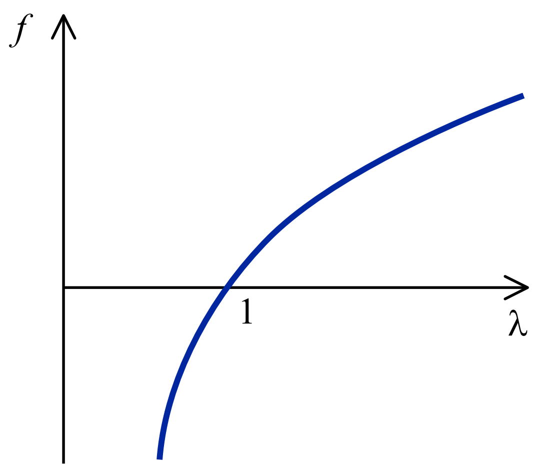
$$w(\lambda) = c(\lambda^2 + 1/\lambda^2 - 2)$$

where c=constant and the **bond stretch** is

$$\lambda = \frac{|\eta + \xi|}{|\xi|}$$

hence the **bond force** is

$$f(\lambda) = \frac{2c}{|\xi|}(\lambda - 1/\lambda^3)$$



- Can also include dependence on bond length:

$$w(\lambda,\xi) = c(\lambda^2 + 1/\lambda^2 - 2)g(|\xi|)$$

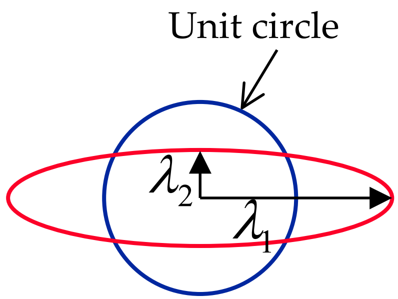
Prototype microelastic membrane under homogeneous deformation

- In a homogeneous deformation, the prototype microelastic membrane material with (bond) micropotential

$$w(\lambda,\xi)=c(\lambda^2+1/\lambda^2-2)g(|\xi|)$$

leads to the bulk strain energy density defined by

$$W=\frac{1}{2}\int_R w(\lambda,\xi)dV, \qquad \lambda=\frac{|F\xi|}{|\xi|}, \qquad [F]=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$



which comes out to

$$W(\lambda_1,\lambda_2)=\frac{\pi hcR}{2}\left(\lambda_1^2+\lambda_2^2+\frac{2}{\lambda_1\lambda_2}-4\right), \qquad R=\int_0^\delta rg(r)dr$$

This is a special case of the Blatz-Ko material

$$W(\lambda_1,\lambda_2,\lambda_3)=\frac{\pi hcR}{2}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\frac{2}{\lambda_1^2\lambda_2^2\lambda_3^2}-4\right)$$

in plane stress.

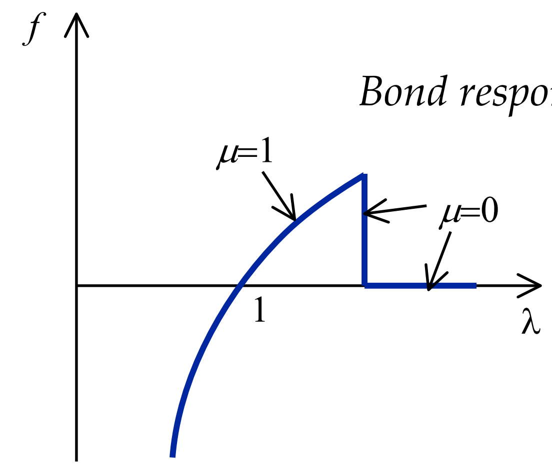
λ is bond stretch.
λ_1,λ_2 are principal stretches of F.

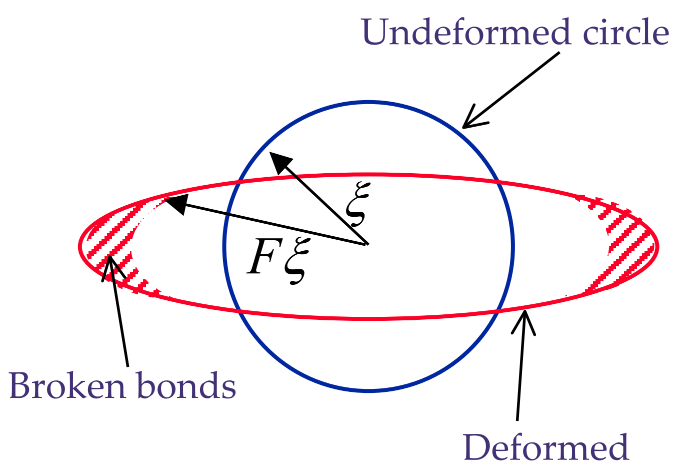
Microelastic membranes with damage

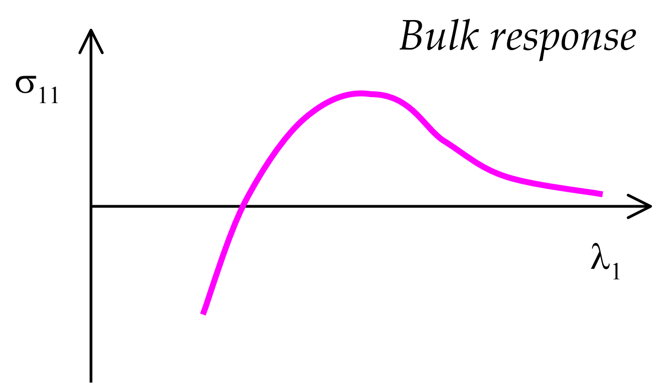
- Prototype material with bond breakage:

$$f(\lambda, x, t) = \frac{2c}{|\xi|}(\lambda - 1/\lambda^3)\mu(x,t)$$

where μ changes irreversibly from 1 to 0 when the bond breaks.



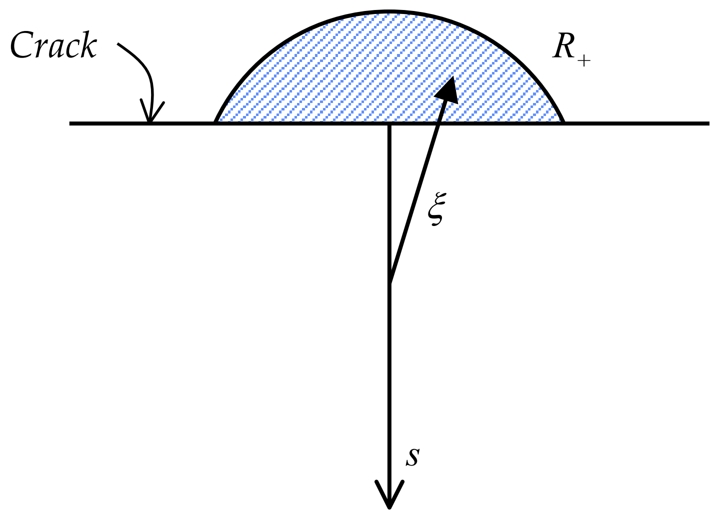


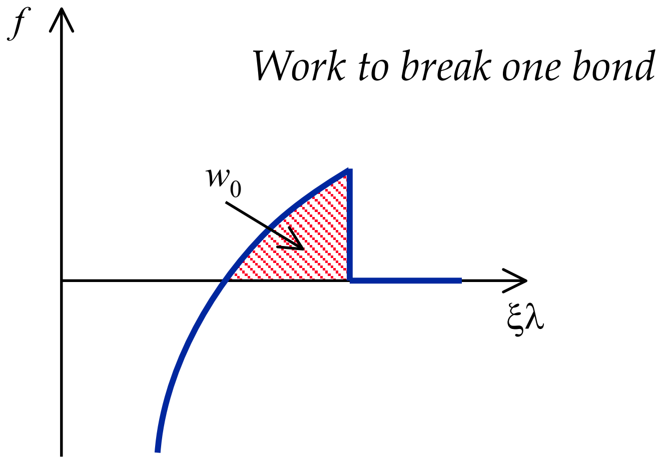


Energy required to advance a crack

- Adding up the work needed to break all bonds across a line yields the energy release rate:

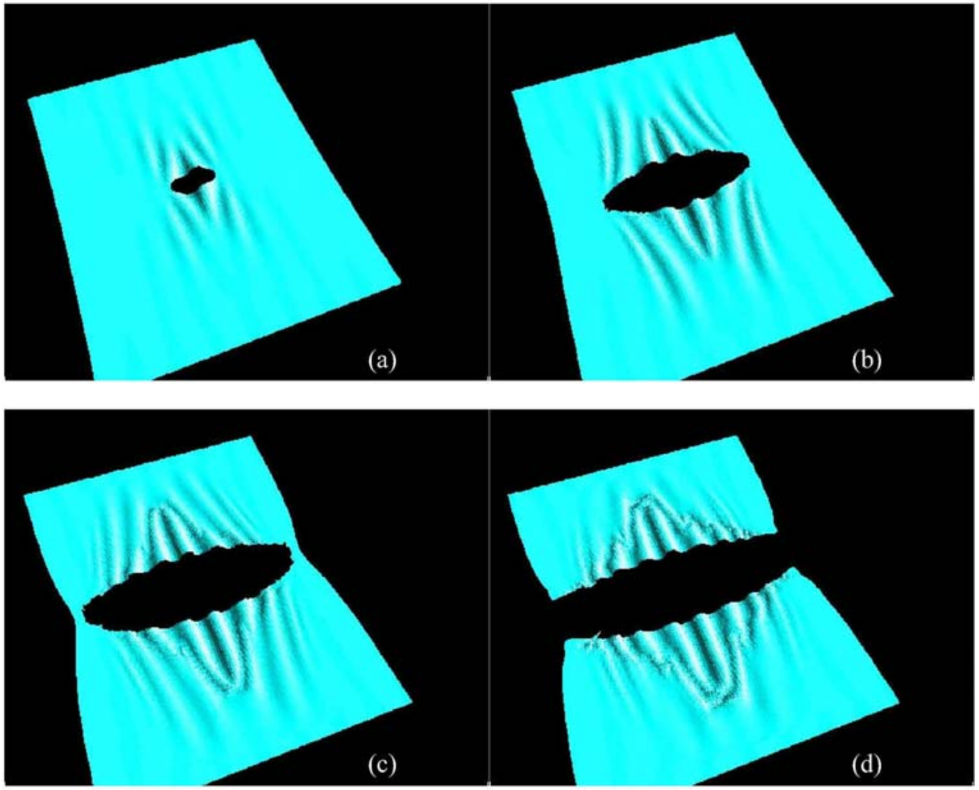
$$G = 2h\int_0^{\delta}\int_{R_+} w_0\,dA\,ds$$





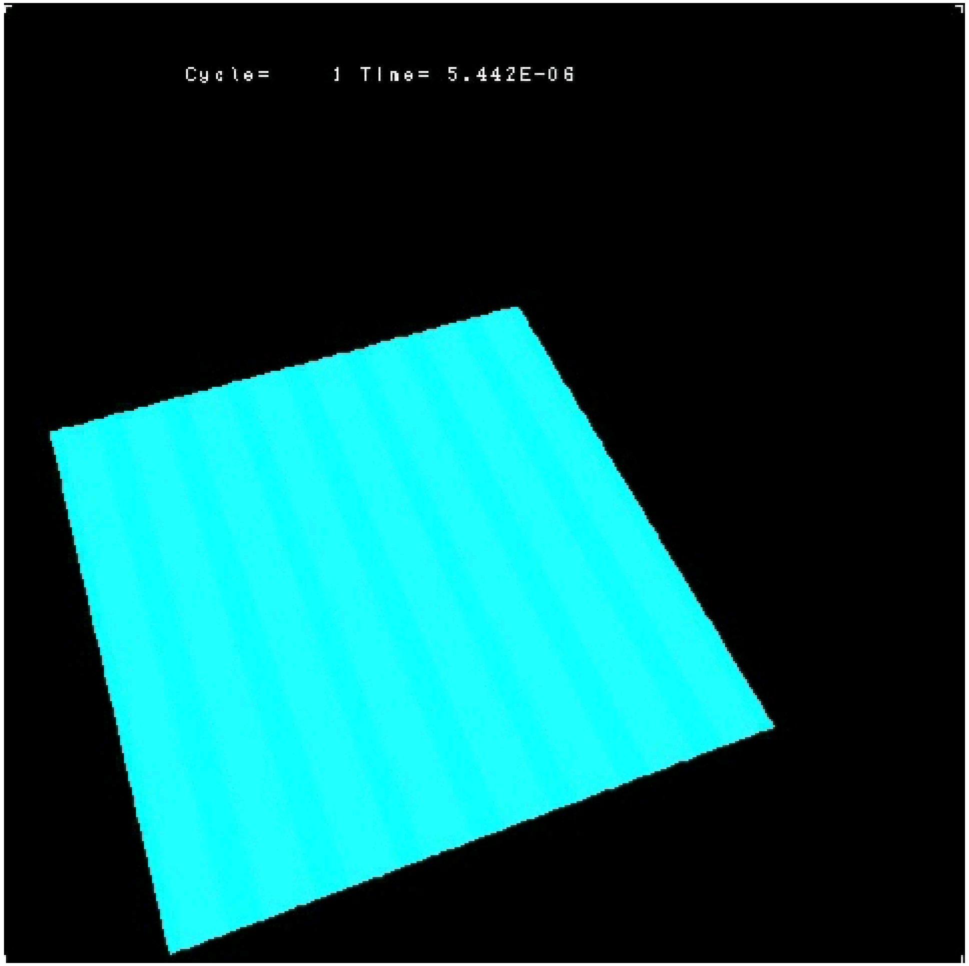
Example:
Tearing of a membrane

- Wrinkles appear due to compressive strains parallel to the crack*.



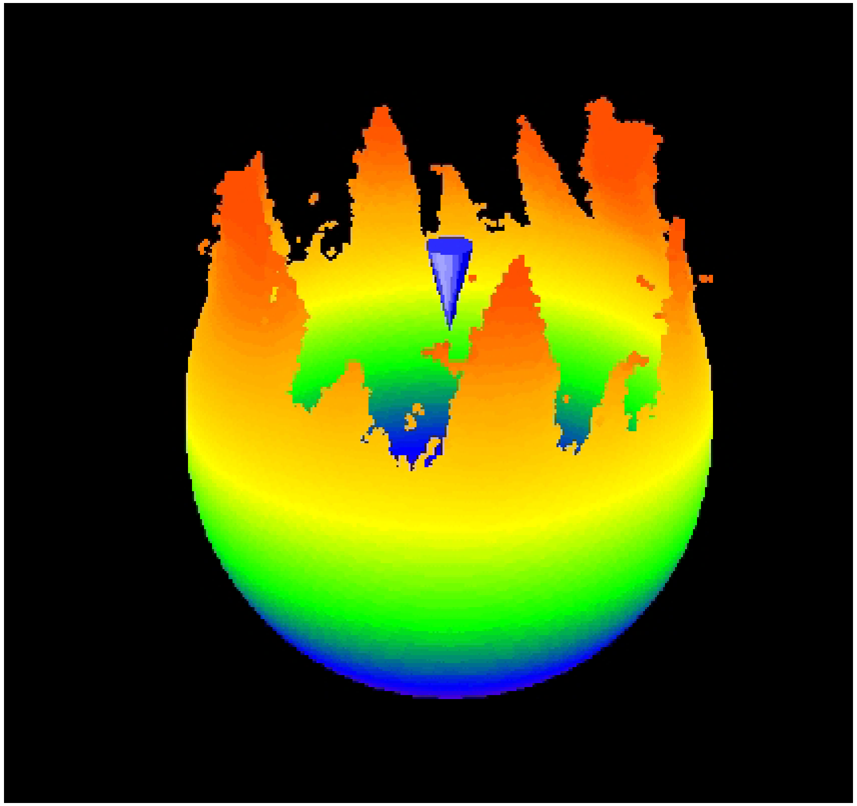
*Also see Haseganu and Steigmann, *Computational Mechanics* (1994) for numerical model of wrinkling.

Example:
Tearing of a membrane (Emu animation)



Example: Balloon pop

- Fragment strikes a pressurized spherical membrane.



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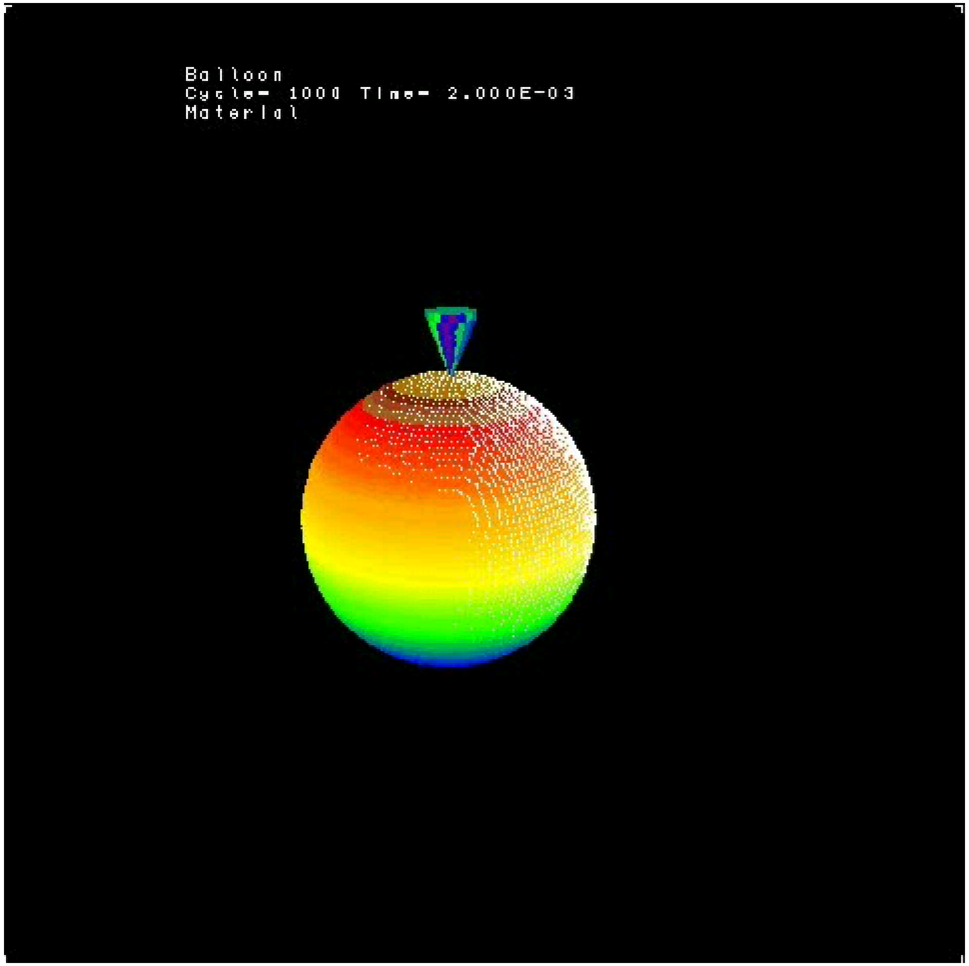

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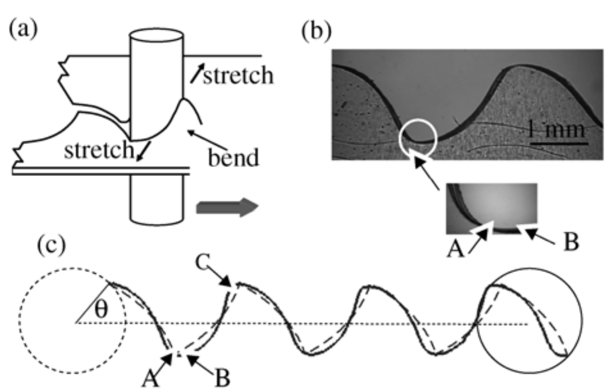
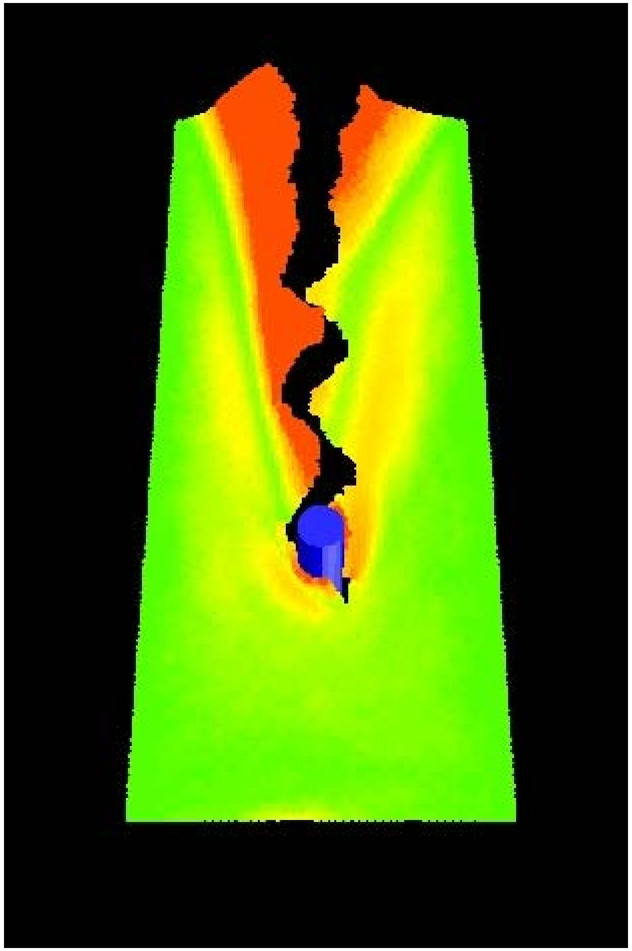
Example:
Oscillatory crack growth in a membrane

- Blunt tool cuts through a microelastic membrane.
- Off-center notch nucleates the crack.
- Oscillations involve friction.

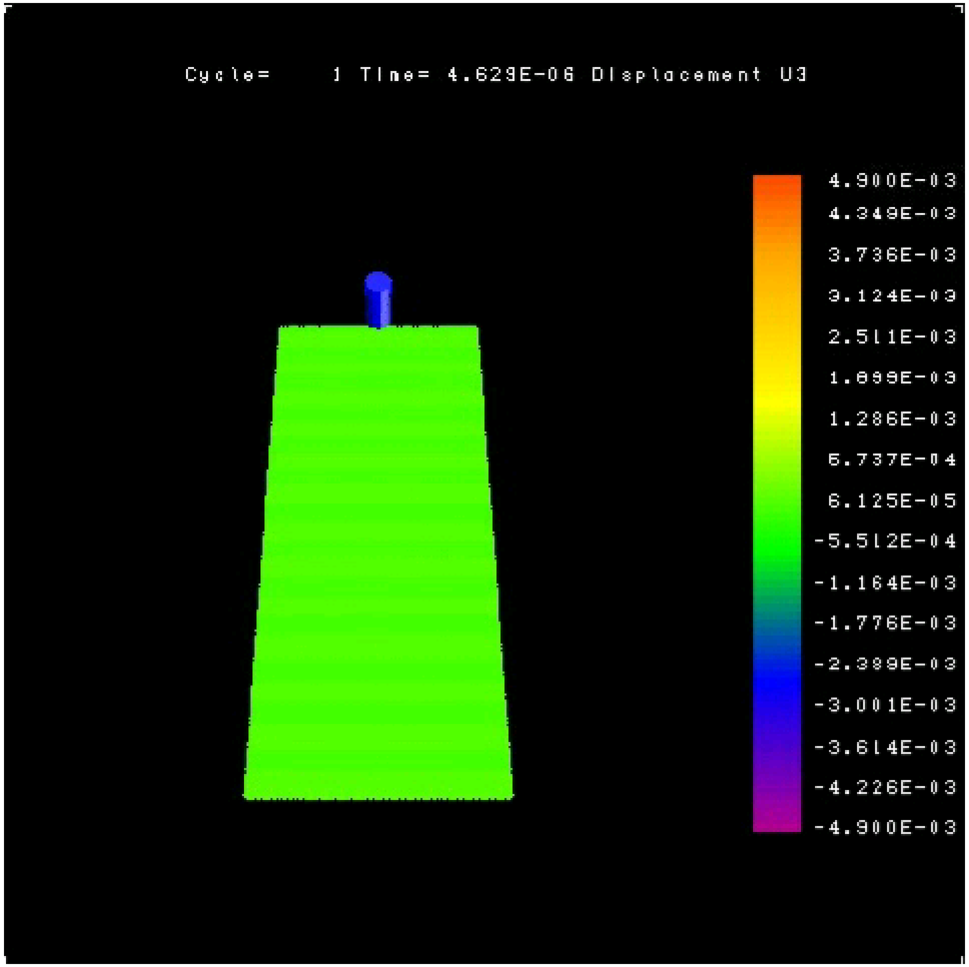


Experimental data of
Ghatak & Mahadevan, *Physical Review Letters* **91** (2003) 215507-1;
also see:
Roman et. Al., *Comptes Rendus Mechanique* **331** (2003) 811.



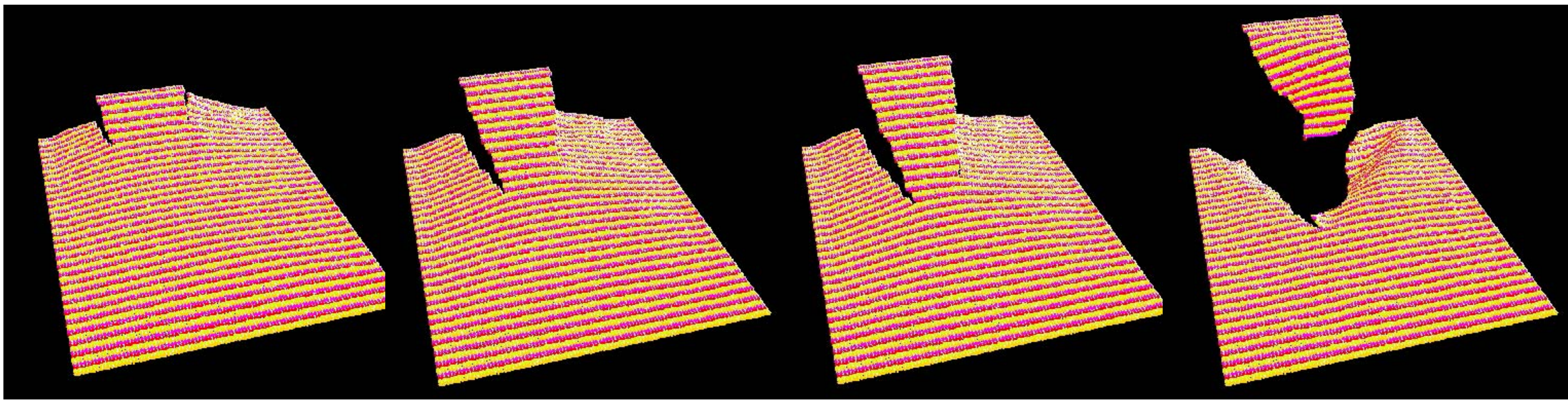
Peridynamic model

Example:
Oscillatory crack growth (Emu animation)

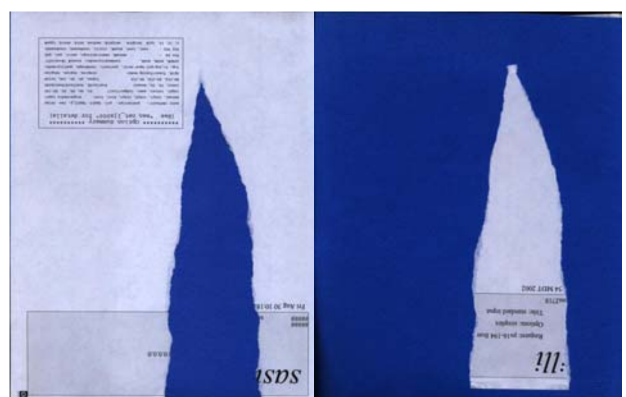


Example:
Tearing of a sheet

- Pull upward on part of a free edge – other 3 edges are fixed.



“Experimental data”



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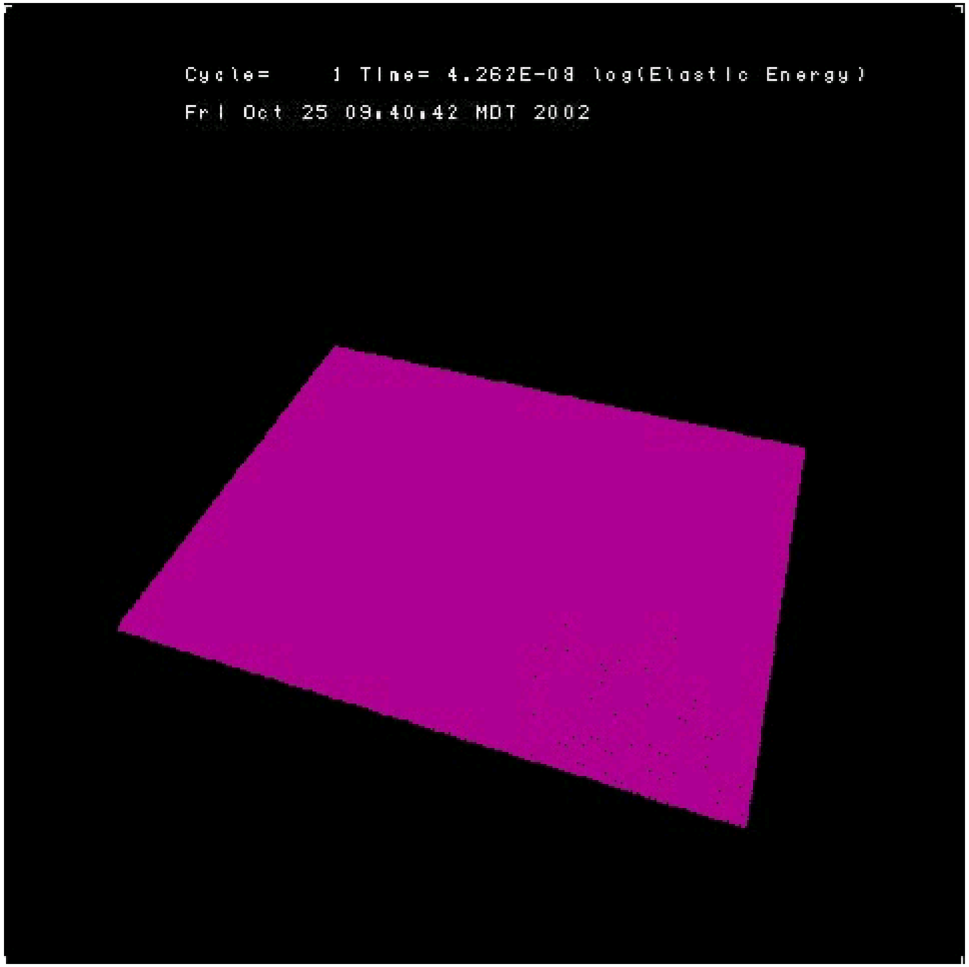

Summary

- The peridynamic model is intended to generalize the classical theory to include discontinuities, especially cracks.
- Constitutive modeling, including damage, takes place at the bond level.
  - Bond response implies a bulk response.
- Fracture occurs spontaneously and can involve complex patterns of crack growth.
- For further information:
  - www.sandia.gov/emu/emu.htm
  - Forthcoming paper in *International Journal of Non-Linear Mechanics*